Context-invariant and local quasi hidden variable (qHV) modelling versus contextual and nonlocal HV modelling

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Abstract

For all the joint von Neumann measurements on a D-dimensional quantum system, we present the specific example of a context-invariant quasi hidden variable (qHV) model, proved in [Loubenets, J. Math. Phys. 56, 032201 (2015)] to exist for each Hilbert space. In this model, a quantum observable X is represented by a variety of random variables satisfying the functional condition required in quantum foundations but, in contrast to a contextual model, each of these random variables equivalently models X under all joint von Neumann measurements, regardless of their contexts. This, in particular, implies the specific local qHV (LqHV) model for an N-qudit state and allows us to derive the new exact upper bound on the maximal violation of 2x...x2-setting Bell-type inequalities of any type (either on correlation functions or on joint probabilities) under N-partite joint von Neumann measurements on an N-qudit state. For d=2, this new upper bound coincides with the maximal violation by an N-qubit state of the Mermin-Klyshko inequality. Based on our results, we discuss the conceptual and mathematical advantages of context-invariant and local qHV modelling.

key words: qHV modelling, nonclassicality, contextuality, quantum nonlocality, Bell-type inequalities
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1 Introduction

In quantum theory, the interpretation of von Neumann [1] measurements via the probability model[1] of the classical statistical mechanics, that is, in terms of random variables and probability measures on a single measurable space[2] $(\Omega, \mathcal{F}_\Omega)$, is referred to as *a hidden variable (HV) model.* As it is well known, for all quantum observables on a Hilbert space of a dimension $\dim \mathcal{H} \geq 3$, there does not [6] exist a noncontextual HV model where random variables obey the functional subordination inherent to quantum observables. However, for all observables and states on an arbitrary Hilbert space, there does exist [7] a contextual HV model where this functional subordination is satisfied. For details and references, see sections 1, 2 in [8].

Under HV modelling [9] of the probabilistic description of a quantum correlation scenario, noncontextuality is generally interpreted as locality while contextuality – as *quantum nonlocality* conjectured by Bell [10] due to his result [11] on impossibility of a local HV (LHV) description of spin measurements of two parties on the two-qubit singlet. Bell argued [10] that the Einstein-Podolsky-Rosen (EPR) paradox [12] should be resolved specifically due to the violation of *locality* under multipartite quantum measurements.

The mathematical result of Bell in [11] can be, however, conditioned by either of at least two mathematical alternatives: (i) the dependence of a random variable at one site not only on an observable measured at this site but also on measurement settings and outcomes at the other sites; (ii) non-positivity of a scalar measure ν modelling the singlet state. From the physical point of view, a choice between these two mathematical alternatives corresponds to a choice between (i) *nonlocality* and (ii) *nonclassicality.* The latter results in violation of the "classical realism" embedded into the probability model of the classical statistical mechanics just via probability measures. Bell chose the first alternative and conjectured [11] *quantum nonlocality.*

However, as we stressed in section 3 of [9], though both, *the EPR locality and Bell's locality*, correspond to the manifestation of the physical principle of *local action* under nonsignaling multipartite measurements, but – the EPR locality, described in [12] as "without in any way disturbing" systems and measurements at other sites, is a general concept, not in any way associated with the use of some specific

[1]In the mathematical physics literature, this probability model is often named after Kolmogorov. However, in the probability theory literature, the term "Kolmogorov model" is mostly used [2] for the Kolmogorov probability axioms [3]. These axioms hold for a measurement of any nature, in particular, quantum, see also our discussion in [4, 5].

[2]Here, Ω is a set and $\mathcal{F}_\Omega$ is an algebra of subsets of Ω.

mathematical formalism, whereas Bell's locality (as it is specified in [10, 11]) constitutes the manifestation of locality specifically in the HV frame. As a result, though Bell' locality implies the EPR locality, the converse is not true – the EPR locality *does not need* [9] to imply Bell' locality.

On the other hand, if to view quantum nonlocality only as the manifestation of quantum entanglement, then it is not clear why, for some nonseparable quantum states [13, 9], all quantum correlation scenarios with $S_n \leq S_n^{(0)}$ settings at each n-th site admit the LHV description while whenever a number of settings $S_n > S_n^{(0)}$, then the LHV description of correlation scenarios with such settings is not possible.

All this *questions*[3] the conceptual meaning of Bell's "quantum nonlocality".

Note that, in quantum information theory, a nonlocal nonseparable quantum state is defined explicitly via nonexistence for all joint von Neumann measurements on this state of a single LHV model formulated in ([17]), therefore, via violation by this state of some Bell-type inequality[4].

Analyzing the probabilistic description of a general correlation scenario, we have introduced in [19] a new notion – the notion of *a local quasi hidden variable (LqHV) model,* which is specified in terms of "local" random variables on a measure space $(\Omega, \mathcal{F}_\Omega, \nu)$ but, in this triple, a normalized measure ν can be real-valued. In a LqHV model, all scenario joint probabilities are reproduced via nonnegative values of a real-valued measure ν and all the product expectations – via the qHV (classical-like) average of the product of the corresponding "local" random variables.

We proved [19] that the Hilbert space description of every quantum correlation scenario admits LqHV modelling. Moreover, we showed [5] that the probabilistic description of *every* nonsignaling correlation scenario does also admit LqHV modelling.

Note that the proved [19, 5] possibility of LqHV modelling of each quantum correlation scenario corresponds just to the second above alternative – *nonclassicality* which was, however, disregarded by Bell.

Based on our results [19, 5] on the probabilistic modelling of nonsignaling correlation scenarios, we also formulated [5] a new general probability model, *the quasi-classical probability model,* where the measure theory structure inherent to the probability model of the classical statistical mechanics is preserved but measures ν on a measurable space

[3]See also in [14, 15, 16].

[4]For the general framework on Bell-type inequalities, see [18].

$(\Omega, \mathcal{F}_\Omega)$ can be real-valued. In this new probability model, all observed joint probabilities are reproduced only via *nonnegative* values of real-valued measures ν. In a quantum case, we refer to this model as *a quasi hidden variable (qHV) model.*

Furthermore, we have recently proved by theorem 3 in [8] that, for each Hilbert space, the Hilbert space description of all the joint von Neumann measurements admits a *context-invariant* qHV model – a model of a completely new type where a quantum observable X can be represented on a measurable space $(\Omega, \mathcal{F}_\Omega)$ by a variety of random variables satisfying the functional condition required in quantum foundations, but, in contrast to a contextual[5] HV model, each of these random variables *equivalently* models X under all joint von Neumann measurements, regardless of their measurement contexts. For $\dim \mathcal{H} \geq 4$, the HV version of a context-invariant model for all quantum observables and states cannot [8] exist.

The proved [8] existence for each Hilbert space of a context-invariant qHV model, in particular, implies the existence (proposition 3 in [8]) for each N-partite quantum state of a LqHV model – the notion introduced in [19].

In the present paper, we present the specific example (section 2) of a context-invariant qHV model [8] reproducing in measure theory terms the Hilbert space description of all the joint von Neumann measurements on a D-dimensional quantum system. This, in turn, implies the specific LqHV model for each N-qudit state and allows us to derive (section 3) the new exact upper bound $\min\{d^{\frac{N-1}{2}}, 3^{N-1}\}$ on the maximal violation of $2 \times \cdots \times 2$-setting[6] Bell-type inequalities of any type, either on correlation functions or on joint probabilities, under N-partite joint von Neumann measurements on an N-qudit state. In section 4, we summarize the main results and stress the mathematical advantages of context-invariant and local qHV modelling.

2 Context-invariant qHV modelling

For the Hilbert space $\mathbb{C}^D$, $D \geq 2$, let us introduce the specific example of *a context-invariant qHV model* [8] reproducing the Hilbert space description of all the joint von Neumann measurements.

Denote by $\mathfrak{X}_D$ the set of all quantum observables on $\mathbb{C}^D$ and by

[5] In a contextual model, a quantum observable can be also modelled by a variety of random variables but which of these random variables represents an observable X under a joint von Neumann measurement depends *specifically on a context* of this joint measurement, i. e. on other compatible quantum observables measured jointly with X.

[6] This notation means [18] that two observables are measured at each of N sites.

Λ the set of all real-valued functions $\lambda : \mathfrak{X}_D \to \cup_{X \in \mathfrak{X}_D} \mathrm{sp}X$ with values $\lambda(X) \equiv \lambda_X$ in the spectrum $\mathrm{sp}X$ of the corresponding quantum observable X.

Let $\pi_{(X_1,...,X_n)} : \Lambda \to \mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n$ be the canonical projection on Λ :

$$\begin{aligned} \pi_{(X_1,...,X_n)}(\lambda) &= (\pi_{X_1}(\lambda), ..., \pi_{X_n}(\lambda)), \\ \pi_X(\lambda) &= \lambda_X \in \mathrm{sp}X, \end{aligned} \tag{1}$$

and $\mathcal{A}_\Lambda$ be the algebra of all the cylindrical subsets of the form

$$\begin{aligned} \pi^{-1}_{(X_1,...,X_n)}(F) &= \{\lambda \in \Lambda \mid (\pi_{X_1}(\lambda), ..., \pi_{X_n}(\lambda)) \in F\}, \\ F &\subseteq \mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n, \end{aligned} \tag{2}$$

for all collections $\{X_1, ..., X_n\} \subset \mathfrak{X}_D, n \in \mathbb{N}$, of quantum observables on $\mathbb{C}^D$. For a Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ and an arbitrary quantum observable X, the random variable $\varphi \circ \pi_X \neq \pi_{\varphi(X)}$.

According to proposition 1 and relation (20) in [8], to every qudit state ρ on $\mathbb{C}^D$, there corresponds ($\rho \overset{\mathfrak{R}}{\mapsto} \mu_\rho$) a unique normalized real-valued measure $\mu_\rho : \mathcal{A}_\Lambda \to \mathbb{R}$, $\mu_\rho(\Lambda) = 1$, satisfying the representation

$$\begin{aligned} &\mu_\rho(\pi^{-1}_{(X_1,...,X_n)}(F)) \\ = &\frac{1}{n!} \sum_{(x_1,...,x_n) \in F} \mathrm{tr}[\rho\{\mathrm{P}_{X_1}(x_1) \cdot \ldots \cdot \mathrm{P}_{X_n}(x_n)\}_{\mathrm{sym}}] \end{aligned} \tag{3}$$

for all subsets $F \subseteq \mathrm{sp}X_1 \times \cdots \times \mathrm{sp}X_n$ and all finite collections $\{X_{1,....,}X_n\} \subset \mathfrak{X}_D$ of quantum observables on $\mathbb{C}^D$. Here, $\mathrm{P}_{X_i}(\cdot)$ is the spectral (projection-valued) measure of an observable X_i with eigenvalues $\{x_i\}$, possibly degenerate, and the notation $\{\cdot\}_{sym}$ means the sum arising due to the symmetrization of the operator product standing in $\{\cdot\}$ with respect to all permutations of its factors.

If $\rho_j \overset{\mathfrak{R}}{\mapsto} \mu_{\rho_j}$, $j = 1, ..., m$, then

$$\sum \alpha_j \rho_j \overset{\mathfrak{R}}{\mapsto} \sum \alpha_j \mu_{\rho_j}, \quad \alpha_j > 0, \ \sum \alpha_j = 1. \tag{4}$$

For mutually commuting quantum observables $X_1, ..., X_n$, $n \in \mathbb{N}$, representation (3) implies

$$\begin{aligned} &\mathrm{tr}[\rho\{\mathrm{P}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{P}_{X_n}(B_n)\}] \\ = \ &\mu_\rho\left(\pi^{-1}_{X_1}(B_1) \cap \cdots \cap \pi^{-1}_{X_n}(B_n)\right) \end{aligned} \tag{5}$$

for all subsets $B_i \subseteq \mathrm{sp}X_i$, $i = 1, ..., n$.

For each quantum observable $X \in \mathfrak{X}_D$, denote by

$$\begin{aligned}
[\pi_X] &= \{f_{X,\theta} \mid \theta \in \Theta_X\}, \quad \text{where } f_{X,\theta_0} \equiv \pi_X, \qquad (6)\\
f_{X,\theta} &= \phi_X^{(\theta)} \circ \pi_{Y_\theta}, \quad \phi_X^{(\theta)} \circ Y_\theta = X, \quad \phi_X^{(\theta)} : \mathbb{R} \to \mathbb{R}, \quad Y_\theta \in \mathfrak{X}_D,
\end{aligned}$$

the non-empty set of random variables, satisfying the spectral correspondence rule $f_{X,\theta}(\Lambda) = \mathrm{sp}X$. The index set Θ_X depends only on properties of a quantum observable X. If $X_1 \neq X_2$, then $[\pi_{X_1}] \cap [\pi_{X_2}] = \varnothing$, for the proof, see appendix D in [8] .

Let $\mathfrak{F}_{\mathfrak{X}_D} = \cup_{X \in \mathfrak{X}_D} [\pi_X]$ be the union of all the disjoint sets $[\pi_X]$, $X \in \mathfrak{X}_D$, of random variables on Λ and $\Psi : \mathfrak{F}_{\mathfrak{X}_D} \to \mathfrak{X}_D$ be the mapping

$$\Psi(f) := X, \quad f \in [\pi_X] \subset \mathfrak{F}_{\mathfrak{X}_D}, \quad X \in \mathfrak{X}_D, \qquad (7)$$

specifying the correspondence between random variables in the set $\mathfrak{F}_{\mathfrak{X}_D}$ and quantum observables on $\mathbb{C}^D$. From (6) it follows that, for each Borel function $\varphi : \mathbb{R} \to \mathbb{R}$ and every random variable $f_{X,\theta} \in [\pi_X]$, we have:

$$\begin{aligned}
\varphi \circ f_{X,\theta} &= (\varphi \circ \phi_X^{(\theta)}) \circ \pi_{Y_\theta} \qquad (8)\\
&\in [\pi_{(\varphi \circ \phi_X^{(\theta)}) \circ Y_\theta}] = [\pi_{\varphi(X)}],
\end{aligned}$$

so that $\varphi \circ f_{X,\theta}$ is one of random variables, representing on Λ the quantum observable $\varphi(X)$. Therefore, the mapping (7) satisfies the functional condition (8) required in quantum foundations, see, for example, section 1.4 in [7].

Due to definition (6) of sets $[\pi_X]$, $X \in \mathfrak{X}_D$, and lemma 2 in [8], the relation

$$\begin{aligned}
& \mu_\rho \left(\pi_{X_1}^{-1}(B_1) \cap \cdots \cap \pi_{X_n}^{-1}(B_n) \right) \qquad (9)\\
= \; & \mu_\rho \left(f_{X_1,\theta_1}^{-1}(B_1) \cap \cdots \cap f_{X_n,\theta_n}^{-1}(B_n) \right)
\end{aligned}$$

holds for arbitrary random variables $f_{X_1,\theta_1} \in [\pi_{X_1}], ..., f_{X_n,\theta n} \in [\pi_{X_n}]$, representing on Λ the corresponding quantum observables.

From (5), (9) it follows that, for each finite collection $\{X_1, ..., X_n\}$, $n \in \mathbb{N}$, of mutually commuting observables on $\mathbb{C}^D$, all the von Neumann joint probabilities $\mathrm{tr}[\rho\{\mathrm{P}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{P}_{X_n}(B_n)\}]$, $B_i \subseteq \mathrm{sp}X_i$, admit the representation

$$\begin{aligned}
& \mathrm{tr}[\rho\{\mathrm{P}_{X_1}(B_1) \cdot \ldots \cdot \mathrm{P}_{X_n}(B_n)\}] \qquad (10)\\
= \; & \mu_\rho \left(\pi_{X_1}^{-1}(B_1) \cap \cdots \cap \pi_{X_n}^{-1}(B_N) \right)\\
= \; & \mu_\rho \left(f_{X_1,\theta_1}^{-1}(B_1) \cap \cdots \cap f_{X_n,\theta_n}^{-1}(B_n) \right), \quad \forall \theta_i \in \Theta_{X_i},
\end{aligned}$$

which holds for arbitrary random variables $f_{X_1,\theta_1} \in [\pi_{X_1}], ..., f_{X_n,\theta n} \in [\pi_{X_n}]$, representing on Λ the corresponding quantum observables. For the product expectations, representation (10) immediately implies

$$\begin{aligned} & \mathrm{tr}[\rho(X_1 \cdot \ldots \cdot X_n)] \\ = & \int_\Lambda \pi_{X_1}(\lambda) \cdot \ldots \cdot \pi_{X_n}(\lambda)\mu_\rho(\mathrm{d}\lambda) \\ = & \int_\Lambda f_{X_1,\theta_1}(\lambda) \cdot \ldots \cdot f_{X_n,\theta_n}(\lambda)\mu_\rho(\mathrm{d}\lambda), \quad \forall \theta_i \in \Theta_{X_i}. \end{aligned} \tag{11}$$

Both representations, (10) and (11), are *context-invariant* in the sense that, regardless of a context of a joint von Neumann measurement, into the right-hand sides of these representations, each of random variables $f_{X,\theta} \in [\pi_X]$ modelling a quantum observable X can be *equivalently* substituted. Therefore, the set $[\pi_X]$, defined by relation (6), constitutes the class of random variables equivalently representing X under all joint von Neumann measurements. The correspondence $[\pi_X] \leftrightarrow X$, specified by (1), (6) and (7) is one-to-one.

The context-invariant representations (10) and (11) reproduce the Hilbert space description of all the joint von Neumann measurements on qudits in measure theory terms – i. e. via random variables, satisfying the functional condition (8) generally required in quantum foundations, and the real-valued normalized measures μ_ρ on the measurable space $(\Lambda, \mathcal{A}_\Lambda)$. Therefore, these representations constitute the specific example of *a context-invariant qHV model* proved in [8] to exist for each Hilbert space.

Remark. Due to representation (5), for each state ρ on $\mathbb{C}^D$, all the elements $\omega_\rho(s, U) = \langle s|U\rho U^+|s\rangle = \mathrm{tr}[\rho\{U^+|s\rangle\langle s|U\}] \geq 0, \sum_s \omega_\rho(s, U) = 1$, of the quantum state tomogram - the notion introduced in [20], constitute the corresponding nonnegative values of the real-valued normalized measure μ_ρ. Here, U is an unitary operator on $\mathbb{C}^D$ and $\{|s\rangle, s = 1, ..., D\}$ is an orthonormal basis in $\mathbb{C}^D$.

3 Local qHV (LqHV) modelling

Consider now local quasi hidden variable (LqHV) modelling [19] of the Hilbert space description of N-partite joint von Neumann measurements on a state $\rho_{d,N}$ on $(\mathbb{C}^d)^{\otimes N}$.

According to proposition 3 in [8], for each N-qudit state $\rho_{d,N}$, the context-invariant qHV representation (10) implies the following LqHV

model:

$$\begin{aligned} & \mathrm{tr}[\rho_{d,N}\ \{\mathrm{P}_{X_1}(B_1)\otimes\cdots\otimes\mathrm{P}_{X_N}(B_N)\}] \qquad (12)\\ = & \ \mu_{\rho_{d,N}}\left(\pi^{-1}_{\widetilde{X}_1}(B_1)\cap\cdots\cap\pi^{-1}_{\widetilde{X}_N}(B_N)\right)\\ = & \int\limits_{\Lambda}\chi_{\pi^{-1}_{\widetilde{X}_1}(B_1)}(\lambda)\cdot\ldots\cdot\chi_{\pi^{-1}_{\widetilde{X}_N}(B_N)}(\lambda)\ \mu_{\rho_{d,N}}(\mathrm{d}\lambda),\quad B_n\subseteq\mathrm{sp}X_n,\end{aligned}$$

specified in terms of the measure space $(\Lambda,\mathcal{A}_\Lambda,\mu_{\rho_{d,N}})$ with the real-valued normalized measure $\mu_{\rho_{d,N}}$ given by relation (3) and the "local" random variables

$$\begin{aligned} \pi_{\widetilde{X}_n}(\lambda),\quad \widetilde{X}_n &= \mathbb{I}_{\mathbb{C}^d}\otimes\cdots\otimes\mathbb{I}_{\mathbb{C}^d}\otimes X_n\otimes\mathbb{I}_{\mathbb{C}^d}\otimes\cdots\otimes\mathbb{I}_{\mathbb{C}^d}, \qquad (13)\\ X_n &\in \mathfrak{X}_d,\quad n=1,...,N,\end{aligned}$$

where each $\pi_{\widetilde{X}_n}$ corresponds by (1) to a quantum observable X_n measured at n-th site. In (12), $\chi_A(\lambda)$ is the indicator function of a set $A\in\mathcal{A}_\Lambda$, i. e. $\chi_A(\lambda)=1$, if $\lambda\in A$, and $\chi_A(\lambda)=0$, if $\lambda\notin A$.

Let us now apply the LqHV representation (12) for the evaluation under N-partite joint von Neumann measurements on a state $\rho_{d,N}$ of the maximal quantum violation $\Upsilon^{(\rho_{d,N})}_{2\times\cdots\times2}$ of $2\times\cdots\times2$-setting Bell-type inequalities of any type, either on correlation functions or on joint probabilities. In mathematical terms, this state parameter is, in general, defined by Eq. (52) in [19].

Denote by $X_n^{(i_n)}$, $i_n=1,2, n=1,....,N$, two quantum observables, with eigenvalues $\{x_n^{(i_n)}\}$, possibly, degenerate, *projectively* measured at each n-th of N sites under a $2\times\cdots\times2$-setting correlation scenario on a state $\rho_{d,N}$. Let $\tau^{(\rho_{d,N})}_{2\times\cdots\times2}(\cdot|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)})$ be a real-valued normalized measure in a LqHV model for this scenario. In the LqHV terms, the expression for $\Upsilon^{(\rho_{d,N})}_{2\times\cdots\times2}$ follows from the general relations (40) - (42) in [19] and reads

$$\Upsilon^{(\rho_{d,N})}_{2\times\cdots\times2}=\sup_{\substack{X_n^{(i_n)},i_n=1,2,\\ n=1,...,N}}\ \inf\left\|\tau^{(\rho_{d,N})}_{2\times\cdots\times2}(\cdot|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)})\right\|_{var}, \qquad (14)$$

where $\left\|\tau^{(\rho_{d,N})}_{2\times\cdots\times2}\right\|_{var}$ is the total variation norm[7] of a measure $\tau^{(\rho_{d,N})}_{2\times\cdots\times2}$ and infimum is taken over all possible LqHV models.

By restricting, in view of (3), the measure $\mu_{\rho_{d,N}}$ on $\mathcal{A}_\Lambda$ to the subalgebra of cylindrical subsets of the form $\pi^{-1}_{(\widetilde{X}_1^{(1)},\widetilde{X}_1^{(2)},...,\widetilde{X}_N^{(1)},\widetilde{X}_N^{(2)})}(F)$, $F\subseteq\Omega$, where

$$\Omega=\mathrm{sp}X_1^{(1)}\times\mathrm{sp}X_1^{(2)}\times\cdots\times\mathrm{sp}X_N^{(1)}\times\mathrm{sp}X_N^{(2)},$$

[7] On this notion, see, for example, section 3 in [19].

and slightly modifying the resulting distribution, for a $2\times\cdots\times 2$-setting correlation scenario on a state $\rho_{d,N}$, we come due to (12) to the following LqHV model:

$$\begin{aligned} &\mathrm{tr}[\rho_{d,N}\{\mathrm{P}_{X_1^{(i_1)}}(B_1^{(i_1)})\otimes\cdots\otimes\mathrm{P}_{X_N^{(i_N)}}(B_N^{(i_N)})\}] \qquad (15)\\ =\ &\sum_{\omega\in\Omega}\Big(\prod_{n=1,\dots,N}\chi_{B_n^{(i_n)}}(x_n^{(i_n)})\Big)\ \nu_{2\times\cdots\times2}^{(\rho_{d,N})}(\omega|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)}), \end{aligned}$$

for where $\omega=(x_1^{(1)},x_1^{(2)},...,x_N^{(1)},x_N^{(2)})$, $B_n^{(i_n)}\subseteq \mathrm{sp}X_n^{(i_n)}$, $i_n=1,2$, and the real-valued distribution $\nu_{2\times\cdots\times2}^{(\rho_{d,N})}$ is specified in Appendix.

By Eq. (27), the total variation norm

$$\begin{aligned} &\left\|\nu_{2\times\cdots\times2}^{(\rho_{d,N})}(\cdot|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)})\right\|_{var} \qquad (16)\\ =\ &\sum_{\omega\in\Omega}\left|\nu_{2\times\cdots\times2}^{(\rho_{d,N})}(\omega|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)})\right| \end{aligned}$$

of the distribution $\nu_{2\times\cdots\times2}^{(\rho_{d,N})}$ is upper bounded as

$$\left\|\nu_{2\times\cdots\times2}^{(\rho_{d,N})}(\cdot|X_1^{(1)},X_1^{(2)},...,X_N^{(1)},X_N^{(2)})\right\|_{var}\leq d^{\frac{N-1}{2}}. \qquad (17)$$

From relations (14) and (17) it follows that, for N-partite joint von Neumann measurements on a state $\rho_{d,N}$, *the maximal* $2\times\cdots\times2$-*setting Bell violation*

$$\Upsilon_{2\times\cdots\times2}^{(\rho_{d,N})}\leq d^{\frac{N-1}{2}}. \qquad (18)$$

Taking also into account theorem 4 in [19], which implies $\Upsilon_{2\times\cdots\times2}^{(\rho_{d,N})}\leq 3^{N-1}$, we finally derive

$$\Upsilon_{2\times\cdots\times2}^{(\rho_{d,N})}\leq \min\{d^{\frac{N-1}{2}},3^{N-1}\}. \qquad (19)$$

This new upper bound essentially improves the result following for $S=2$ from the general exact upper bound (62) in [19].

For $d=2$ and $N=2$, the upper bound in (19) is equal to $\sqrt{2}$ and, therefore, coincides with the well-known result on the maximal quantum violation of Bell-type inequalities on correlation functions and joint probabilities for two settings and two outcomes per site – the result following from the analysis of Fine in [21] and the Tsirelson maximal quantum violation bound [22] for the Clauser-Horne-Shimony-Holt (CHSH) inequality.

For $d=2$ and an arbitrary $N>2$, the upper bound in (19) is equal to $2^{\frac{N-1}{2}}$ and, hence, coincides with the maximal violation by an N-qubit state of the Mermin-Klyshko inequality. Therefore, under N-partite joint von Neumann measurements on an N-qubit state,

the Mermin-Klyshko inequality gives the maximal violation not only among all correlation $2 \times \cdots \times 2$-setting Bell-type inequalities (as it is proved in [23]) but also among $2 \times \cdots \times 2$-setting Bell-type inequalities of any type.

4 Conclusions

We have presented the specific example of a context-invariant quasi hidden variable (qHV) model reproducing due to representations (10), (11) the Hilbert space description of all the joint von Neumann measurements on qudits via random variables satisfying the functional condition (8) required in quantum foundations and the real-valued measures on the measurable space $(\Lambda, \mathcal{A}_\Lambda)$. In this context-invariant qHV model, a quantum observable X is represented by the whole class $[\pi_X]$ of random variables, but, in contrast to a contextual HV model, each of these random variables equivalently models X under all joint von Neumann measurements, regardless of their measurement contexts. The correspondence $X \leftrightarrow [\pi_X]$, specified by Eqs. (1), (6) and (7), is one-to-one.

For each N-qudit state, the context-invariant qHV model (10) implies the specific LqHV model (15) which allows us to derive the new exact upper bound (19) on the maximal quantum violation of $2 \times \cdots \times 2$-setting Bell-type inequalities of any type, either on correlation functions or on joint probabilities, under N-partite joint von Neumann measurements on an N qubit state. Specified for $d = 2$, this new upper bound coincides with the maximal violation by an N qubit state of the Mermin-Klyshko inequality. This proves that, under N-partite joint von Neumann measurements on an N-qubit state, the Mermin-Klyshko inequality gives the maximal quantum violation among *all* possible $2 \times \cdots \times 2$-setting Bell-type inequalities, not necessarily the correlation ones. Note that, in the nonlocal HV frame, such quantum calculations are not possible in principle.

From the conceptual point of view, the new type [8, 19] of mathematical modelling of joint quantum measurements in measure theory terms, *context-invariant and local qHV modelling,* corresponds to the second alternative disregarded by Bell [10, 11] – *nonclassicality*, resulting in violation of the "classical realism" embedded into the probability model of the classical statistical mechanics via probability measures. Context-invariant and local qHV modelling is free from the conceptual inconsistencies inherent to Bell's concept of "quantum nonlocality" discussed in Introduction. Moreover, as our results in [19] and in the present paper demonstrate, this new type of probabilistic modelling is fruitful for quantum calculations.

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5 Appendix

Let us first consider the bipartite case $N = 2$. For simplicity of notations, denote by X_i, $i = 1, 2$, observables measured by Alice and by Y_k, $k = 1, 2$. measured by Bob. For this case, the values of the real-valued distribution $\nu_{2\times2}^{(\rho_{d,2})}(\cdot|X_1, X_2, Y_1, Y_2)$, standing in (15), are defined as

$$\begin{aligned}
&2\nu_{2\times2}^{(\rho_{d,2})}(x_1, x_2, y_1, y_2 \mid X_1, X_2, Y_1, Y_2) \qquad (20)\\
&= \alpha_{X_1}^{(+)}(x_1|y_1, y_2)\alpha_{X_2}^{(+)}(x_2|y_1, y_2)\ \mathrm{tr}[\rho_{d,2}\ \{\mathbb{I}_{\mathbb{C}^d} \otimes \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(+)}\}]\\
&- \alpha_{X_1}^{(-)}(x_1|y_1, y_2)\alpha_{X_2}^{(-)}(x_2|y_1, y_2)\mathrm{tr}[\rho_{d,2}\ \{\mathbb{I}_{\mathbb{C}^d} \otimes \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(-)}\}],
\end{aligned}$$

where (i) the notation $Z^{(\pm)}$ means the positive operators, decomposing a self-adjoint operator $Z = Z^{(+)} - Z^{(-)}$ and satisfying the relation $Z^{(+)}Z^{(-)} = Z^{(-)}Z^{(+)} = 0$, (ii) the probability distributions $\alpha_{X_i}^{(\pm)}(\cdot|y_1, y_2)$, $i = 1, 2$, are defined via the relation

$$\begin{aligned}
&\mathrm{tr}[\rho_{d,2}\{\mathrm{P}_{X_i}(x_i) \otimes \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(\pm)}\}] \qquad (21)\\
&= \alpha_{X_i}^{(\pm)}(x_i|y_1, y_2))\mathrm{tr}\rho_{d,2}\{\mathbb{I}_{\mathbb{C}^d} \otimes \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(\pm)}\}].
\end{aligned}$$

From (20) and (21) it follows that, for the distribution $\nu_{2\times2}^{(\rho_{d,2})}$, the total variation norm

$$\begin{aligned}
&\left\|\nu_{2\times2}^{(\rho_{d,2})}(\cdot|X_1, X_2, Y_1, Y_2)\right\|_{var} \qquad (22)\\
&\equiv \sum_{\omega\in\Omega}\left|\nu_{2\times\cdots\times2}^{(\rho_{d,N})}(x_1, x_2, y_1, y_2|X_1, X_2, Y_1, Y_2)\right|\\
&\leq \frac{1}{2}\left\|\sum_{y_1,y_2}|\ \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}|\right\|_{\mathbb{C}^d},
\end{aligned}$$

where $|\ \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}|$ is the absolute value operator

$$\begin{aligned}
&|\{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}| \qquad (23)\\
&= \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(+)} + \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}^{(-)}.
\end{aligned}$$

Calculating $|\{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}|$, we find

$$\begin{aligned}
&\sum_{y_1,y_2}|\ \{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\mathrm{sym}}| \qquad (24)\\
&= \sum_{k_1,k_2}|\alpha_{k_1,k_2}|\left(|\phi_{Y_1}^{(k_1)}\rangle\langle\phi_{Y_1}^{(k_1}|\ +\ |\phi_{Y_2}^{(k_2)}\rangle\langle\phi_{Y_2}^{(k_2}|\right),
\end{aligned}$$

where $\phi_Y^{(k)}$, $k = 1, ..., d$, are orthonormal eigenvectors of an observable Y and $\alpha_{k_1,k_2} = \langle \phi_{Y_1}^{(k_1)} | \phi_{Y_2}^{(k_2)} \rangle$. Substituting (24) into (22) and taking into account that $\sum_{k_i} |\alpha_{k_1,k_2}| \leq \sqrt{d}$, $i = 1, 2$, we finally derive

$$\left\| \nu_{2\times 2}^{(\rho_{d,2})}(\cdot | X_1, X_2, Y_1, Y_2) \right\|_{var} \leq \sqrt{d}. \tag{25}$$

For $N > 2$, the real-valued distribution

$$\nu_{2\times\cdots\times 2}^{(\rho_{d,N})}(\omega | X_1^{(1)}, X_1^{(2)}, ..., X_N^{(1)}, X_N^{(2)}) \tag{26}$$

in (15) is similar by its construction to distribution (20) with the replacement of $\{\mathbb{I}_{\mathbb{C}^d} \otimes \frac{1}{2}\{\mathrm{P}_{Y_1}(y_1)\mathrm{P}_{Y_2}(y_2)\}_{\text{sym}}^{(\pm)}\}$ by the N-partite tensor product of identity operator $\mathbb{I}_{\mathbb{C}^d}$ and $(N-1)$ factors of the form $\frac{1}{2}\{\mathrm{P}_{X_n^{(1)}}(x_n^{(1)})\mathrm{P}_{X_n^{(2)}}(x_n^{(2)})\}_{\text{sym}}^{(\pm)}$ at each n-th of $(N-1)$ sites. As a result,

$$\left\| \nu_{2\times\cdots\times 2}^{(\rho_{d,N})} \right\|_{var} \leq d^{\frac{N-1}{2}}. \tag{27}$$

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